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Some Results on Non-Negative Matrices

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Recently Mirsky and Farahat proposed the problem of characterizing the class of doubly stochastic matrices for which the least number of permutation matrices necessary to represent it as a convex sum has a prescribed value. It is shown that this number can be related to the number of eigenvalues of modulus one. The problem of similarity of doubly stochastic matrices is also treated. Finally, the question of transitivity of powers of sets of functions on the first n positive integers into itself is treated by defining a corresponding incidence matrix and examining its powers.

I. Introduction

In 1946 [1] [3] Birkhoff proved that any doubly stochastic (d.s.) matrix with non-negative entries is a convex combination of permutation matrices. We recall that a matrix is d.s. if every row and column sum is 1. Let Ω_n denote the polyhedron of non-negative d.s. matrices and as noted in [5] dim $\Omega_n=(n-1)^2$ and hence any $A\epsilon\Omega_n$ can be written as a convex combination of at most $(n-1)^2+1$ permutation matrices. Mirsky and Farahat in a recent paper [6] suggested an investigation of the minimum number $\beta(A)$ of permutation matrices necessary to represent $A\epsilon\Omega_n$ as a convex combination. In section 2 we obtain an inequality relating $\beta(A)$ to $h(A)$, the number of characteristic roots of A of absolute value 1. In section 2 we also study a problem of the similarity of two matrices in Ω_n.

In 3 we discuss conditions on $A\geq 0$ that are implied by the equality det $(I-A)=\prod_{i=1}^{n}(1-a_{ii})$.

In 4 we obtain an inequality for the difference between the maximal characteristic roots of two matrices A and B satisfying $B\geq A\geq 0$, where $B\geq A$ means $b_{ij}\geq a_{ij}$ for all i,j. In certain cases our inequality yields better bounds than those obtained by applying the results in [2] or [7].

In 5 we discuss the question of transitivity of sets of functions on the set of integers $\{1, \ldots, n\}$ into itself by matrix methods.

2. Doubly Stochastic Matrices

A precise statement of the result relating $h(A)$ and $\beta(A)$ is contained in theorem 3. We prove two preliminary results of some interest in themselves. Recall that $A\geq 0$ is *reducible* if there exists a permutation matrix P such that

$$PAP'=\begin{pmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{pmatrix}, \tag{1}$$

where A_{11} and A_{22} are square matrices. It is clear that if $A\epsilon\Omega_n$ then (1) implies that $A_{21}=0$. It follows that in case $A\epsilon\Omega_n$ a permutation matrix P may be chosen so that PAP' is the direct sum of irreducible d.s. matrices and 1-square matrices.

THEOREM 1. *If $S\epsilon\Omega_n$ and S is irreducible, then $h(S)$ is a divisor of n.*

PROOF. By the classical Perron-Frobenius theorem, more recently reproved in [8], there exists a permutation matrix P such that

$$PSP'=\begin{bmatrix}0_1 & S_1 & 0 & \ldots & 0\\ . & 0_2 & S_2 & \ldots & 0\\ . & & & & \\ . & & & & S_{h-1}\\ S_h & 0 & & \ldots & 0_h\end{bmatrix}, \tag{2}$$

where 0_i is a principal square matrix of zeros and $h(S)=h$. Let $r(X)$ $(c(X))$ denote the number of rows (columns) of X. Clearly since $S\epsilon\Omega_n$ each S_i is d.s. and $r(S_i)$ is the sum of the elements of S_i as is $c(S_i)$. Hence,

$$r(S_i)=c(S_i)=c(0_{i+1})=r(0_{i+1})=r(S_{i+1})=c(S_{i+1}). \tag{3}$$

Thus every S_i is of the same order and $h(S)$ is a divisor of n. We next obtain a relation between $\beta(S)$ and $\beta(S_i)$ where $S=S_1\dot{+}\ldots\dot{+}S_m$ and $\dot{+}$ denotes the direct sum.

THEOREM 2. *If $S_i\epsilon\Omega_{n_i}$, $i=1, \ldots, m$, $n=\sum_{i=1}^{m}n_i$ and $S=S_1\dot{+}\ldots\dot{+}S_m$, then*

$$\beta(S)\leq\sum_{i=1}^{m}\beta(S_i)-m+1. \tag{4}$$

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[3] Figures in brackets indicate the literature references at the end of this paper.

PROOF. First we prove by induction on $r+s$ that if

$$A=\sum_{i=1}^{r} \theta_i P_i, \qquad B=\sum_{j=1}^{s} \varphi_j Q_j,$$

$$0<\theta_1\leq\theta_2\leq \ldots \leq\theta_r, 0<\varphi_1\leq\varphi_2 \ldots \leq\varphi_s, \sum_{i=1}^{r} \theta_i=1=\sum_{j=1}^{s} \varphi_j,$$

P_i, Q_j are permutation matrices, $i=1, \ldots, r$, $j=1, \ldots, s$, then $A\dot{+}B$ is in the convex hull of no more than $r+s-1$ permutation matrices of the form $P_i\dot{+}Q_j$. When $r+s=2$, then $A=P_1$, $B=Q_1$, and $A\dot{+}B=P_1\dot{+}Q_1$. Suppose for convenience that $\phi_1\leq\phi_1$. Then

$$A\dot{+}B=\theta_1(P_1\dot{+}Q_1)+(1-\theta_1)\left[\left(\sum_{i=2}^{r}\frac{\theta_i}{1-\theta_1}Px\right)\right.$$

$$\left.\dot{+}\left(\frac{\varphi_1-\theta_1}{1-\theta_1}Q_1+\sum_{j=2}^{s}\frac{\varphi_j}{1-\theta_1}Q_j\right)\right], \tag{5}$$

and $\sum_{j=2}^{r}\frac{\theta_j}{1-\theta_1}=1$, and

$$\frac{\varphi_1-\theta_1}{1-\theta_1}+\sum_{j=2}^{s}\frac{\varphi_j}{1-\theta_1}=\frac{1}{1-\theta_1}\sum_{j=1}^{s}\varphi_j-\frac{\theta_1}{1-\theta_1}=1.$$

Thus $A\dot{+}B=\theta_1(P_1\dot{+}Q_1)+(1-\theta_1)R$ where $R\epsilon\Omega_n$ is the direct sum of two d.s. matrices, the first involving $r-1$ permutation matrices and the second involving s permutation matrices. We apply the induction to R to conclude that $\beta(R)\leq r+s-2$. Hence $\beta(A\dot{+}B)\leq r+s-1$. Then an obvious induction on m yields

$$\begin{aligned}\beta(S)&=\beta(S_1\dot{+}\ldots\dot{+}S_m)\\ &\leq\beta(S_1\dot{+}\ldots\dot{+}S_{m-1})+\beta(S_m)-1\\ &\leq\sum_{j=1}^{m-1}\beta(S_j)-m+2+\beta(S_m)-1\\ &=\sum_{j=1}^{m}\beta(S_j)-m+1.\end{aligned}$$

THEOREM 3. *If $S\epsilon\Omega_n$ and S is irreducible, then*

$$\beta(S)\leq h\left(\frac{n}{h}-1\right)^2+1. \tag{6}$$

where $h=h(S)$.

PROOF. We first observe that if P and Q are permutation matrices then $\beta(PSQ)=\beta(S)$ for any $S\epsilon\Omega_n$. By theorem 1 let $n=qh$ where q is an integer and let R be the permutation matrix with 1 in positions

$$(q+i, i) \quad \bmod n.$$

Then from (2) we easily compute that

$$PSP'R=S_1\dot{+}\ldots\dot{+}S_{h-1}\dot{+}S_h, \tag{7}$$

and hence by theorem 2

$$\beta(PSP'R)=\beta(S)\leq\sum_{i=1}^{h}\beta(S_i)-h+1.$$

But by the Birkhoff result quoted in the introduction we have that

$$\beta(S_i)\leq(q-1)^2+1.$$

Hence,

$$\beta(S)\leq h[(q-1)^2+1]-h+1=h\left(\frac{n}{h}-1\right)^2+1.$$

Actually the above techniques may be applied to yield better estimates on $\beta(S)$ than that given by a direct application of (6). There are several alternatives for each of the matrices S_i in (7): S_i is reducible; S_i is irreducible with $h(S_i)>1$; S_i is irreducible with $h(S_i)=1$; i.e., S_i is *primitive*. These situations are covered by the following two theorems.

THEOREM 4. *If $S\epsilon\Omega_n$ and P is such a permutation matrix that $PSP'=T_1\dot{+}\ldots\dot{+}T_k$ where each T_j is an n_j-square matrix either irreducible or with $n_j=1$, then*

$$\beta(S)\leq\sum_{j=1}^{k}\left(\frac{n_j}{h(T_j)}-1\right)^2 h(T_j)+1.$$

THEOREM 5. *If $S\epsilon\Omega_n$ and P and Q are such permutation matrices that*

$$PSQ=S_1\dot{+}\ldots\dot{+}S_m, \tag{8}$$

where each S_j is an n_j-square matrix either primitive or with $n_j=1$, then

$$\beta(S)\leq\sum_{j=1}^{m}(n_j-1)^2+1.$$

We describe the procedure for obtaining the form (8) for S: First reduce S to the direct sum of irreducible or 1-square d.s. matrices. Each of these may then be reduced to the form (2) which in turn is reduced to a direct sum of irreducible or 1-square matrices. The process is repeated until every matrix which appears is either 1-square or primitive. We illustrate this with the following example: Let $S\epsilon\Omega_8$ be given by

$$S=\begin{pmatrix}0_4 & J_4\\ A & 0_4\end{pmatrix},$$

where $A=\begin{pmatrix}0_2 & I_2\\ J_2 & 0_2\end{pmatrix}$, 0_t and I_t are t-square zero and identity matrices respectively, and $J_t\epsilon\Omega_t$ is the t-square matrix all of whose entries are $1/t$. We verify directly that no coordinate subspace is held invariant by S and hence S is irreducible. We first note that $S^2=J_4\dot{+}J_4$ and hence the characteristic roots of S^2 are 1 and 0. Hence the characteristic

roots of S are 1, -1, and 0 with appropriate multiplicities. On the other hand, 1 is simple by the Perron-Frobenius result and $\mathrm{tr}(S)=0$. Hence the characteristic roots of S must be 1, -1, and 0 (six times) and $h(S)=2$. Thus S is already in the form (2) and we bring S by a permutation to the form $J_4\dot{+}A$ and by a further one-sided permutation to $J_4\dot{+}I_2\dot{+}J_2=J_4\dot{+}I_1\dot{+}I_1\dot{+}J_2$.

Thus by theorem 2,

$$\beta(S)\leq\beta(J_4)+\beta(I_1)+\beta(I_1)+\beta(J_2)-4+1. \tag{9}$$

A direct application of theorem 3 to J_4 yields $\beta(J_4)\leq 10$ and (9) implies that

$$\beta(S)\leq 11. \tag{10}$$

Of course it is obvious by inspection that $\beta(J_4)=4$ and hence (9) gives $\beta(S)\leq 5$. Note that a direct application of theorem 3 yields $\beta(S)\leq 19$ and the bound given by the dimension of Ω_8 is $\beta(S)\leq 50$. As a matter of fact, $\beta(S)=4$.

Recently one of the present authors and A. J. Hoffman independently proved that if $S\epsilon\Omega_n$ then there exists a permutation σ such that $\prod_{i=1}^{n} s_{i\sigma(i)}\geq 1/n^n$. It follows that the permanent of S satisfies the following inequality:

$$\text{per }(S)=\sum_{\sigma}\prod_{i=1}^{n} s_{i\sigma(i)}\geq 1/n^n$$

for $S\epsilon\Omega_n$. Since

$$S=\sum_{i=1}^{\beta(S)}\alpha_i P_i,$$

where P_i are permutation matrices and $\alpha_i\geq 0$, $\sum_{i=1}^{\beta(S)}\alpha_i=1$, we see that

$$\text{per }(S)\geq(1/\beta(S))^{n-1}.$$

It follows that we may apply the estimates of theorems 3, 4, 5 to obtain a lower bound for S that in some cases is better than $1/n^n$. These results are motivated by a conjecture of v. der Waerden [4] concerning the minimum value of the permanent of a matrix in Ω_n.

We next discuss circumstances under which two matrices in Ω_n are similar. If $S\epsilon\Omega_n$ and $XSX^{-1}\epsilon\Omega_n$ it of course is false that either X or X^{-1} must be d.s.; e.g., take $S=I_n$. However if the root 1 of S is simple then we have the following result.

THEOREM 6. *If $S\epsilon\Omega_n$ and S is irreducible and $XSX^{-1}=R\epsilon\Omega_n$, then X is a multiple of a d.s. matrix. Moreover, there exists $Y\epsilon\Omega_n$ such that $YSY^{-1}=R$.*

PROOF. Let J be the n-square matrix all of whose entries are 1, and let $\sigma_i(X)$ be the sum of the entries in the ith row of X. Then $XS=RX$ implies that $XSJ=RXJ$, $XJ=R(XJ)$. Each column of XJ is equal to the n-tuple $(\sigma_1(X), \ldots, \sigma_n(X))=u(X)$. Then $Ru(X)=u(X)$ and since 1 is a simple characteristic root of S and R (S is irreducible), $u(X)$ must be a scalar multiple of the vector e all of whose coordinates are 1. Thus $\sigma_i(X)=\alpha$, $i=1, \ldots, n$, and by a similar argument we easily see that $\sigma_i(X')=\beta$, $i=1, \ldots, n$. But then $\alpha=\beta$ and X is a scalar multiple of a d.s. matrix. Now X commutes with J and thus the characteristic roots of $X+kJ$ are λ_1+kn, $\lambda_2, \ldots, \lambda_n$, where λ_i are the roots of X and λ_1 corresponds to the characteristic vector e. Choose k so that $\lambda_1+kn\neq 0$ and $X+kJ>0$, and let $Y=(\alpha+nk)^{-1}(X+kJ)$. This completes the proof.

3. Triangular Non-Negative Matrices

THEOREM 7. *Let $A\geq 0$ with maximal non-negative characteristic root $r\leq 1$. Then a necessary and sufficient condition that there exists a permutation matrix P such that PAP' is triangular is*

$$\prod_{i=1}^{n}(1-a_{ii})=\det(I_n-A). \tag{11}$$

PROOF. We prove first that if

$$f(t)=\prod_{i=1}^{n}(t-a_{ii})-\det(tI_n-A), \tag{12}$$

then $f(t)\geq 0$ for $t\geq r$. In this argument and subsequently we use the classical result that states that if α is the dominant non-negative characteristic root of a principal submatrix of A, then $\alpha\leq r$ and if A is irreducible then the inequality is strict. Hence, $f(r)=\prod_{i=1}^{n}(r-a_{ii})-\det(rI_n-A)\geq 0$ and the inequality is strict if A is irreducible. To prove that $f(t)\geq 0$ we proceed by induction on n. Now differentiating (12) we have

$$f'(t)=\sum_{i=1}^{n}\left\{\prod_{j\neq i}(t-a_{jj})-\det(tI_{n-1}-A(i|i))\right\}, \tag{13}$$

where $A(i|i)$ is the principal submatrix of A obtained by omitting the ith row and column. For each i the summand in (13) is non-negative by the induction hypothesis (since $t\geq r$ and r is not smaller than the dominant non-negative characteristic root of $A(i|i)$). Thus $f'(t)\geq 0$ and $f(r)\geq 0$ so that $f(1)\geq 0$. Moreover if A were irreducible it would follow that $f(1)\geq f(r)=\prod_{i=1}^{n}(r-a_{ii})>0$.

We complete the proof as follows. Choose a permutation matrix P such that PAP' has the following sequence of blocks down the main diagonal.

$$B_1\dot{+}\ldots\dot{+}B_m=B,$$

where each B_i is 1-square or possibly n_i-square irreducible. We show that the second alternative is impossible and it follows that PAP' is triangular. Clearly (11) implies that

$$\prod_{i=1}^{n}(1-a_{ii})=\prod_{i=1}^{n}(1-b_{ii})=\det(I_n-A)=\det(I_n-B)$$

$$=\prod_{k=1}^{m}\det(I_{n_k}-B_k).$$

Suppose some $B_k=C$ is irreducible. Then by the above argument

$$\prod_{i=1}^{n_k}(1-c_{ii})>\det(I_{n_k}-B_k),$$

and we would conclude that

$$\prod_{k=1}^{m}\det(I_{n_k}-B_k)<\prod_{i=1}^{n}(1-a_{ii}),$$

a contradiction. This completes the proof.

4. Inequalities for the Dominant Characteristic Root

Let $E^{(n)}$ denote the unit n-simplex. If $A\geq 0$ is irreducible, let α be the dominant positive root and let $0<x\epsilon E^{(n)}$ be the unique characteristic vector of A in $E^{(n)}$ corresponding to α. Suppose $R\geq 0$, $R\neq 0$ commutes with A. Then $Ax=\alpha x$, $RAx=\alpha Rx$, $A(Rx/\sigma(Rx))=\alpha(Rx/\sigma(Rx))$ where if $y=(y_1\ \ldots\ y_n)$ then $\sigma(y)=\sum_{i=1}^{n}y_i$. But then the uniqueness of x implies that $Rx/\sigma(Rx)=x$ because $(Rx/\sigma(Rx))\epsilon E^{(n)}$.

THEOREM 8. *Let $B\geq A$ be two non-negative irreducible n-square matrices with dominant positive roots β and α respectively. Let $M=\max_{i,j}(b_{ij}-a_{ij})$, $m=\min_{i,j}(b_{ij}-a_{ij})$ and suppose that R is a non-negative matrix which commutes with either A or B and possesses no zero columns. Then*

$$\frac{m}{\max_i \max_j \frac{r_{ij}}{\sum_{k=1}^{n} r_{kj}}}\leq\beta-\alpha\leq\frac{M}{\min_i \min_j \frac{r_{ij}}{\sum_{k=1}^{n} r_{kj}}}. \tag{14}$$

PROOF. Let $y\epsilon E^{(n)}$, $y>0$, satisfy $Ay=\alpha y$ and let $\mu=(By)_i/y_i=\max_k(By)_k/y_k\geq\beta$. Then we assert that $\beta-\alpha\leq M/y_i$; for $(B-A)y+Ay=By$,

$$(B-A)y+\alpha y=By,$$

$$\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k+\alpha y_i=(By)_i,$$

$$\frac{1}{y_i}\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k+\alpha=\frac{(By)_i}{y_i}=\mu\geq\beta.$$

Now $y\epsilon E^{(n)}$; hence $\sum_{k=1}^{n}(b_{ik}-a_{ik})y_k\leq M$, and $\beta-\alpha\leq M/y_i\leq M/\min_k y_k$.

From the above discussion we know that $y=Ry/\sigma(Ry)$. Consider the function

$$z_k(x)=\frac{\sum_{j=1}^{n} r_{kj}x_j}{\sum_{j=1}^{n}\sum_{i=1}^{n} r_{ij}x_j};\qquad x\epsilon E^{(n)}; \tag{15}$$

minimizing the right-hand side of (15) over all $x\epsilon E^{(n)}$ yields

$$y_k\geq\min_t\frac{r_{kt}}{\sum_{i=1}^{n} r_{it}}.$$

Thus

$$\beta-\alpha\leq M/\min_k y_k\leq M/\min_k\min_t\frac{r_{kt}}{\sum_{i=1}^{n} r_{it}}.$$

A similar computation establishes the lower bound in (14).

Note that R is *any* non-negative matrix commuting with A or B; e.g., any non-negative polynomial in A with no zero columns. We also remark that in practice we compute each column sum of R, divide it into the least element in the column, and take the least such quotient to obtain the denominator of the upper bound in (14). Similarly for the lower bound.

5. A Matrix Condition for Transitivity

Let L_n denote the set of integers $\{1,\ldots,n\}$ and suppose $\sigma_1,\ldots,\sigma_k$ are functions mapping L_n into itself. We define $\mathfrak{A}_k$ to be the set of functions of the form $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$, $1\leq i_j\leq k$. We say that $\mathfrak{A}_p$ is transitive for some p if for any given s and t in L_n there exists $\phi\epsilon\mathfrak{A}_p$ such that $\phi(s)=t$. We define an n-square *incidence matrix* A associated with $\mathfrak{A}_1$ as follows: For each σ_i we define Q_i as the matrix whose (s,t) entry is 1 if $\sigma_i(s)=t$, 0 otherwise; then let $A=\frac{1}{k}\sum_{i=1}^{k}Q_i$. It is clear that $A\geq 0$ and every row sum of A is 1; i.e., A is row stochastic. We remark that if Γ_n denotes the totality of non-negative row stochastic n-square matrices, then an easy argument [6] thows that Γ_n is the convex hull of the set of all matrices with exactly one 1 in each row.

In general let A correspond to $\sigma_1,\ldots,\sigma_k$ and B to $\phi_1,\ldots,\phi_m$. Then the (i,j) entry of AB is positive if and only if the integer i is mapped by some σ_α into p and p is mapped by some ϕ_β into j. That is, $\phi_\beta\sigma_\alpha(i)=j$. Thus we have

THEOREM 9. *A necessary and sufficient condition that $\mathfrak{A}_p$ be transitive is that $A^p>0$.*

We easily obtain two corollaries to this result.

COROLLARY 1. *Let $\mathfrak{A}_1=\{\sigma_1,\ldots,\sigma_k\}$ be a set of functions on $L_n=\{1,\ldots,n\}$ into itself satisfying*

(a) *at least one σ_i has a fixed point,*

(b) *no proper subset of L_n is left invariant (as a set) by every $\sigma_i\epsilon\mathfrak{A}_1$. Then $\mathfrak{A}_p$ is transitive forp $\geq 2n-2$.*

PROOF. Condition (b) implies that the incidence matrix A is irreducible. Condition (a) implies that $\mathrm{tr}(A)>0$ and hence $h(A)=1$. Hence A is primitive and the integer $2n-2$ follows from a result of Holladay and Varga [3].

Corollary 2. *Let* $\mathfrak{A}_1=\{\sigma_1, \ldots, \sigma_k\}$ *be a set of permutations of the integers* $\{1, \ldots, n\}=L_n$ *satisfying*

(a) *there exists an integer* s *in* L_n *and a pair of permutations* σ_i *and* σ_j *such that* $\sigma_j\sigma_i(s)=s$,

(b) *no proper subset of* L_n *is left invariant (as a set) by every* $\sigma_i \epsilon \mathfrak{A}_1$,

(c) n *is odd.*

Then $\mathfrak{A}_p$ *is transitive for* $p \geq 4n-4$.

Proof. If there exists a σ_m and an integer v such that $\sigma_m(v)=v$ then we have the conditions of corollary 1 and $\mathfrak{A}_p$ is transitive for $p\geq 2n-2$ without the condition (c). If not, let A be the incidence matrix and then $\mathrm{tr}(A)=0$, but the sum of all the second order principal subdeterminants of A is not zero. Hence $h(A)$ is 1 or 2. But since $A\epsilon\Omega_n$ we apply theorem 1 to conclude that $h(A)$ divides n. Thus $h(A)=1$ and A is primitive. Now A^2 has a positive element on the main diagonal and $h(A^2)=1$. Thus A^2 is also irreducible and thus primitive so that applying the result in [3] again we have $(A^2)^{2n-2}>0$, completing the proof.

Note that condition (c) is essential as shown by the following example: $\mathfrak{A}_1=\{\sigma_1, \sigma_2\}$ where $\sigma_1=(1\,2\,3\,4)$, $\sigma_2=(1\,4\,3\,2)$ satisfies conditions (a) and (b) of corollary 2. However, $\mathfrak{A}_p$ is not transitive for any p.

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